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Discretization of a Vortex Sheet, with an Example of Roll-Up*

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The point vortex approximation of a vortex sheet in two space dimensions is examined and a remedy for some of its shortcomings is suggested. The approximation is then applied to the study of the roll-up of a vortex sheet induced by an elliptically loaded wing.

INTRODUCTION

The study of the motion of vortex sheets in two-dimensional space is of great importance in a number of practical problems [1, 2, 5], as well as in the design of numerical algorithms [3]. Rosenhead [10] introduced a method of analysis in which the sheet is approximated by an array of point vortices; this method was applied by Westwater [12] to the roll-up problem. Recently, Takami [11] and Moore [8] have shown that this method can produce errors of arbitrarily large magnitude, and thus cast a doubt on its validity. On the other hand, other vortex methods, involving interpolation [4] or a cut-off [3], have recently achieved notable successes in different contexts; one of them has even been proved to be convergent [6]. This discrepancy is of substantial interest, and it is the purpose of the present paper to contribute to its resolution.

It is fairly obvious that a point vortex approximation to a vortex sheet (or for

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that matter, to a continuous vorticity distribution) cannot be taken too literally, since a point vortex induces a velocity field which becomes unbounded, and cannot approximate a bounded field in any reasonable norm. Thus the results of Takami and Moore are understandable. However, we conjecture that as soon as the velocity field of the point vortices is smoothed out and made bounded, i.e., the point character of the point vortices is not taken too literally, the approximation becomes reasonable. Such smoothing occurs in all successful applications, and in fact one may conjecture that the old results of Westwater are better than the new results of Takami and Moore because the limited accuracy of precomputer calculation had a smoothing effect. We intend to present numerical evidence in support of this conjecture in an application to a problem involving the roll-up of a vortex sheet.

Approximation of a Vortex Sheet by a Finite Array of Point Vortices

Consider a vortex sheet whose vorticity is parallel to its length and whose two-dimensional cross section initially lies on the curve C: $x = x(s)$, $y = y(s)$, where x and y are the coordinates of a point and s is the arc length. The vorticity distribution is $\xi = \xi(s)$. The sheet will be moved by the velocity field which it induces; the equations of motion are

$$\frac{\partial x}{\partial t} = -\partial_y \int_C \frac{1}{2\pi} \log |\mathbf{r}(s) - \mathbf{r}(s')| \, \xi(s') \, ds' \tag{1a}$$

$$\frac{\partial y}{\partial t} = \partial_x \int_C \frac{1}{2\pi} \log |\mathbf{r}(s) - \mathbf{r}(s')| \, \xi(s') \, ds' \tag{1b}$$

where $\mathbf{r} = (x, y)$, $|\mathbf{r}| = |x^2 + y^2|^{1/2}$, $x = x(t, s)$, $y = y(t, s)$. In Rosenhead's approximation the sheet is replaced by an array of N point vortices, located at the points $\mathbf{r}_i = (x_i, y_i)$ of C, with vorticities ξ_i, $i = 1,..., N$ whose distribution approximates $\xi(s)$. The motion of these N vorticies is then described by the N ordinary differential equations,

$$\frac{dx_i}{dt} = -\partial_y \sum_{j \neq i} \psi(\mathbf{r} - \mathbf{r}_j) \, \xi_j \,, \tag{2a}$$

$$\frac{dy_i}{dt} = \partial_x \sum_{j \neq i} \psi(\mathbf{r} - \mathbf{r}_j) \, \xi_j \,, \tag{2b}$$

where $\psi = (1/2\pi) \log |\mathbf{r}|$ is the stream function associated with a point vortex (see [1]).

It is readily seen that the right-hand sides of Eqs. (2a), (2b) are rectangle rule approximations to the integrals on the right-hand sides of Eqs. (1), and as such can be expected to approximate the latter well, as long as the derivatives of the integrands are not too large, i.e., as long as $|d\xi/ds|$, $|\partial x/\partial s|$, $|\partial y/\partial s|$ are bounded by some constant K which is small compared to h^{-1}, where h is a typical distance between vortices. When this condition is not satisfied, the approximate balance between the flow due to the vortices on either side of a given vortex will no longer hold because of the unduly large flow produced by a point vortex, and the several vortices will capture each other, i.e., start following complicated paths around each other. Once this process starts at any point, it rapidly spreads throughout the sheet; this type of breakdown was well documented in [2, 8, 11]; see also below.

In summary, as soon as the approximation (2) ceases to be accurate because the numerical parameter h becomes comparable with the intrinsic characteristic lengths of the problem, it also becomes qualitatively unreasonable. This breakdown is analogous to the effect of nonlinear instability in a difference scheme (see [9]). In the case of difference schemes, cures are known: One can sometimes reduce h (at the cost of added computational labor) or, if the region of inaccuracy is initially small, one can introduce an artificial viscosity so designed that its effect is local. The analog of this latter technique is smoothing, which we shall now explain on an example.

Vortex Sheet Induced by an Elliptically Loaded Wing

Consider in particular the vortex sheet initially located on the strip $-a \leqslant x \leqslant a$, with vorticity distribution

$$\xi = 2Ux(a^2 - x^2)^{-1/2}.$$

The significance of this particular configuration is explained in [1]. Clearly the

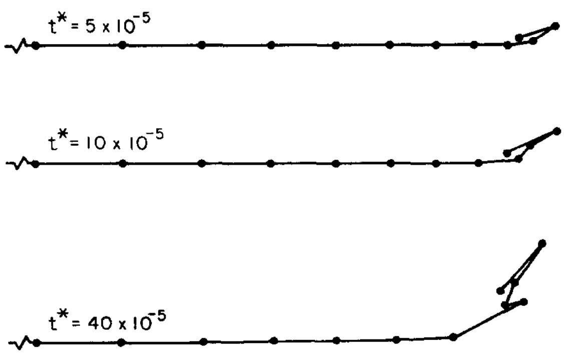

FIG. 1. $M = 50$, $\Delta t^* = 5 \times 10^{-5}$, no cut-off.

vortex point method is likely to fail at the tips of the sheet $x = \pm a$, since at these points $d\xi/dx$ becomes unbounded. A typical breakdown is illustrated in Fig. 1 (which is analogous to the results displayed by Takami [11] and Moore [8]). The strip is divided into $2M$ pieces of equal total vorticity, and a point vortex of strength $\pm 2Ua/M$ is placed at the vorticity centroid of each piece. A dimensionless time $t^* = Ut/a$ is introduced. In the calculation which leads to Fig. 1 we used $M = 50$; equations (2) were integrated using a fourth-order Runge–Kutta method with $\Delta t^* = 5 \times 10^{-5}$. The positions of the last 12 vortices at the extreme right-hand side of the sheet are displayed at times $t^* = 5 \times 10^{-5}$, 10^{-4}, 4×10^{-4}; they are connected in the order in which they were initially placed. Chaos is generated at the tip before the other vortices have had time to move appreciably. The approximation (2) seems to be inapplicable.

Now replace $\psi = (1/2\pi) \log |\mathbf{r}|$ in Eqs. (2) by

$$\psi_\sigma = \begin{cases} (1/2\pi) \log |\mathbf{r}|, & r \geqslant \sigma; \\ (1/2\pi)(|\mathbf{r}|/\sigma) + \text{const} & r < \sigma. \end{cases}$$

σ is a (small) cut-off. The introduction of such a cut-off is analogous to the introduction of a small viscosity which allows the vorticity in a point vortex to diffuse (see [1]). It is artificial rather than real viscosity since its effect is not cumulative; the vorticity is diffused a little, and then spreads no more. If σ is of order a/M^2, only the motion of the few vortices near the tip will be affected. The particular form of ψ_σ was chosen by analogy with the form used in Chorin [3], but in fact almost any form which keeps the velocity field bounded for small enough r seems to be adequate. We again use a Runge–Kutta algorithm, with $\Delta t^* = 5 \times 10^{-5}$, and $\sigma = 6a/M^2$. The results are displayed in Fig. 2 for the times indicated. The spurious

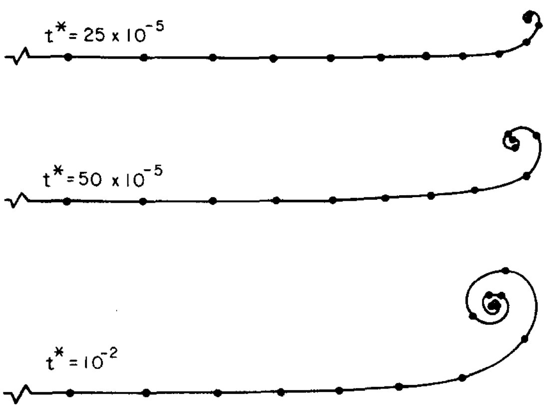

FIG. 2. $M = 50$, $\Delta t^* = 5 \times 10^{-5}$, $\sigma = 6a/M^2$.

motion near the tip has been damped, and the roll-up is proceeding as expected. These results are independent of $\sigma/(a/M^2)$ within a wide range. It is important to note that the introduction of the cut-off does not affect the mutual interaction of distant portions of the sheet; furthermore, the results are not sensitive to the exact choice of ψ_σ for $r < \sigma$.

In choosing a Runge–Kutta integration formula, unevenly spaced vortices and a very small time step, we have followed the example of Takami and Moore, whose aim was to display the breakdown in the approximation. Our aim, of course, was to show that on the contrary, with $\sigma \neq 0$ the roll-up proceeds smoothly under otherwise comparable conditions. If on the other hand, one is interested in the evolution of the sheet, one might as well pick a substantially larger Δt^* (subject to the obvious convergence requirement $U \Delta t \ll a$); it is furthermore reasonable to use a straightforward Euler scheme, which is more economical in terms of computing time and does not introduce errors of larger order than those originating from the replacement of (1) by (2); finally, there is no obvious reason not to use equidistant vortices whose intensity is proportional to the value of ξ at their location. Figures 3 and 4 were obtained under such conditions, with $\Delta t^* = 10^{-2}$. Figure 3 displays the configuration at $t^* = 1$, with $M = 50$, $\sigma = a/M$. These changes effect a saving in computer time, but do not affect the nature of the solution. An effort was made to verify Kaden's result [7] to the effect that the sheet has the form of a spiral with polar equation $r = C(\theta - \theta_0)^{2/3}$ as $\theta \to \infty$. C and θ_0 were determined using the locations of the vortices marked A and B. Vortices 1 through 14 are seen to lie on the resulting spiral with no visible error. Kaden's result has been derived with the help of simplifying assumptions, and is not expected to hold uniformly throughout the spiral [5]. The agreement we obtained is therefore quite satisfactory.

Figure 4 displays the configuration at $t^* = 6.5$, with $\sigma = a/M$. At this time, it is no longer clear that we have an approximation to a sheet. We may as well consider that we have real but small viscosity; the distribution of vorticity in the region of concentrated vorticity is then continuous, and our scheme with cut-off is then

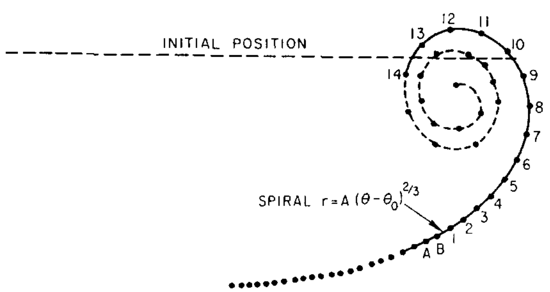

FIG. 3. $\Delta t^* = 10^{-2}$, $M = 50$, $\sigma = a/M$, $t^* = 1$.

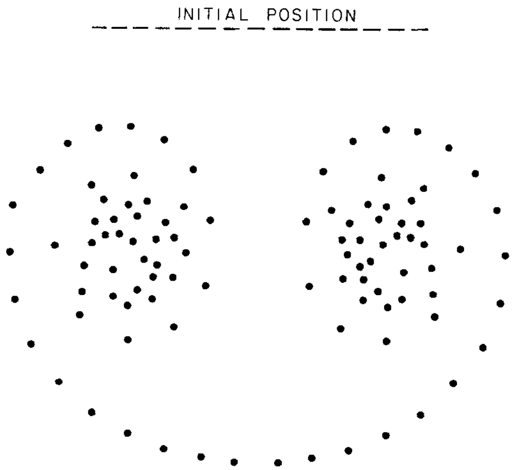

FIG. 4. $\Delta t^* = 10^{-2}$, $M = 50$, $\sigma = 2a/M$, $t^* = 6.5$

known to be valid (see [3, 6]). The results in Figs. 3 and 4 are in fact independent of σ. They are reproduced with only minor differences even when $\sigma = 0$, provided we keep $\Delta t^* = 10^{-2}$; this is consistent with the fact that nonlinear instability has a smaller impact when the time step is increased (see [9]). Thus, one seems to be able to use numerical error as a smoothing mechanism which keeps the point vortex approximation under control.

Conclusion

We have shown that Rosenhead's point vortex approximation is valid and useful, provided the singular character of all the vortices is obviated by some smoothing. This fact explains the discrepancy between the failure of some point vortex approximations and the success of others.

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